Trifactorized Groups

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Factorized groups

A group G is called **factorized**, if

$$G = AB = \{ab \mid a \in A, \ b \in B\}$$

is the product of two subgroups A and B of G.

Main problem.

What can be said about the structure of the factorized group G if the structures of its subgroups A and B are known?

Trifactorized groups

Many proofs about factorized groups finally reduce to the consideration of groups of the form

$$G = AB = AK = BK, A, B \subseteq G, K \lhd G.$$

Definition.

A group G is called **trifactorized** if

$$G = AB = AC = BC$$

for three subgroups A, B and C.

Theorem (O. Kegel 1965)

Let $G = AB = AC = BC$ be a finite trifactorized group, where A and B are nilpotent. (This implies that G is **soluble**.)

1. If C are nilpotent, then G is nilpotent,
2. If C supersoluble, then G is supersoluble.
3. If C belongs to a saturated formation $\mathfrak{F}$ containing all nilpotent groups,
   then also G belongs to $\mathfrak{F}$ (Peterson 1973).

Definition.

A formation $\mathfrak{F}$ is a class of groups closed under epimorphisms which is **residual**, i.e.

if M and N are normal subgroups of the group G such that G/N and G/M are $\mathfrak{F}$-groups, then also $G/(N \cap M)$ is an $\mathfrak{F}$-group.

A formation $\mathfrak{F}$ of finite groups is saturated if the finite group G is an $\mathfrak{F}$-group, **whenever the Frattini factor group $G/\Phi(G)$ is an $\mathfrak{F}$-group.**

Remarks on the proof.

Assume the theorem is false and consider a counterexample $G = AB = AC = BC$ with minimal order.

Show that G is a so-called **primitive group** in the following sense: There exists a uniquely determined minimal normal subgroup M of G such that
$M = C_G(M) = F(G)$ is an elementary-abelian p-group for some prime p.

Show that one of the two subgroups A and B is a p-group, the other a p'-group.

Since the order of A divides the order of $G = BC$, $|A|$ divides $|C|$.
Since the order of B divides the order of $G = AC$, $|B|$ divides $|C|$.
Therefore $|A|\,|B|$ divides $|C|$, so that $G = C$ is an $\mathfrak{F}$-group.

This contradiction proves the theorem.

Groups with minimum condition.

A group G satisfies the **minimum condition for subgroups** ($G \in Min$) if every descending chain of subgroups of G terminates after finitely many steps:

$$G = G_0 \supseteq G_1 \supseteq G_2 \supseteq \ldots\ldots \supseteq G_i \ldots\ldots$$

Then $G_i = G_{i+1}$ for almost all i.

Prüfer groups.

The infinite quasicyclic (Prüfer) p-group $C(p^\infty)$ is isomorphic with the multiplicative group of the complex p^n-roots of unity.

The subgroups of these groups form a chain of subgroups G_i of order p^i

$$1 = G_0 \subset G_1 \subset G_2 \subset \ldots\ldots \subset G_i \subset \ldots\ldots$$

Thus G satisfies Min (but obviously not Max)

Definition.

The **finite residual** $J = J(G)$ of the group G is the intersection of all subgroups of G with finite index

$$J(G) = \bigcap G/N, N \subseteq G, |G : N| < \infty$$

A group G is a **Chernikov group** if

1. $J(G)$ is the direct product of finitely many quasicyclic (Prüfer) p-groups for finitely many primes p,
2. $G/J(G)$ is finite.

(S.N. Chernikov, *1912 Sergej Posad, 1987 Kiev)

Definition.

If $\mathfrak{X}$ is a class of groups, then the group G is **locally** -$\mathfrak{X}$, if every finitely generated subgroup of G is an $\mathfrak{X}$-group.

For example: locally finite, locally soluble, locally nilpotent

Remark.

Chernikov groups appear very often for instance as subgroups of certain matrix groups.

It can be shown that **the Chernikov groups are exactly the locally finite groups with Min (Shunkov).**

Problem.

Let $G = AB = BC = AC$ a trifactorized **Chernikov** group.

Let A and B be locally nilpotent. (This implies that G is soluble)

Let C be an $\mathfrak{X}$-group, where C is a saturated formation containing all locally nilpotent subgroups.

Is G is an $\mathfrak{X}$-group?

"Locally defined" formations of finite soluble groups.

Let f be a function which maps to each prime p a formation $\mathfrak{F}(p)$. Define a formation $\mathfrak{F}$ of finite soluble groups as follows.

$G \in \mathfrak{F}$
$\Longleftrightarrow$
(a) If $\mathfrak{F}(p)$ is empty, then p does not divide the order of G.
(b) If $\mathfrak{F}(p)$ is non-empty and H/K is a p-chief factor of G, then $G/C_G(H/K) \in \mathfrak{F}(p)$.
(H/K is called a p-central chief factor).

Definition.

The class $\mathfrak{F}$ is a formation which is called the
locally defined formation $\mathfrak{F} = LF(f)$.

Theorem of Gaschütz and Lubeseder.

A formation $\mathfrak{F}$ of finite soluble groups is saturated in the above sense, if and only if, it is locally defined.

"Locally defined" and saturated formations of locally finite-soluble groups.

Let f be a function which maps to every prime p a formation $\mathfrak{F}(p)$. Define a formation $\mathfrak{F}$ of locally finite-soluble groups as follows.

$G \in \mathfrak{F}$
$\Longleftrightarrow$
(a) If $\mathfrak{F}(p)$ is empty, then the order of no chief factor of G is an (elementary-abelian) p-group.
(b) If $\mathfrak{F}(p)$ is non-empty and H/K is a p-chief factor of G, then $G/C_G(H/K) \in \mathfrak{F}(p)$.
(H/K is a p-**central chief factor of** G).

Definition.

The class $\mathfrak{F}$ is a formation which is called a
saturated formation of locally finite-soluble groups
$\mathfrak{F} = LF(f)$.

Induction parameters for Chernikov groups.

For a Chernikov group X define the parameter $\Theta(X) = (r, m)$ where

1. $r = r(X)$ is the number of quasicyclic (Prüfer) subgroups in a decomposition of the radicable abelian group $J(X)$ (the **rank** of $J(X)$)
2. $m = m(X) = |X : J(X)|$.

A linear ordering on the set of pairs (r, s) is given by $(r, s) < (r_1, s_1)$ if $r < r_1$ or $r = r_1$ and $s < s_1$.

If U is a subgroup of X, then $\Theta(U) \leq \Theta(X)$.
If $\Theta(U) = \Theta(X)$, then $U = X$.

A General Remark.

If N is a normal subgroup of the factorized group $G = AB$, then the factor group G/N likewise has a factorization

$$G/N = (AN/N)(BN/N)$$

with two epimorphic images AN/N and BN/N of A respectively B.

But **it is very difficult to find subgroups** A_1 of A and B_1 of B such that

$$A_1 B_1 = B_1 A_1$$

is a subgroup of G.

Theorem (B. A., A. Fransman, L. Kazarin 2008, Arch. Math., to appear)

Let $G = AB = AC = BC$ a trifactorized Chernikov group, and let the subgroups A and B be locally nilpotent.

If C belongs to a saturated formation $\mathfrak{F}$ containing all locally nilpotent groups,

then also G is in $\mathfrak{F}$.

In particular, if C is locally nilpotent (locally supersoluble), then G is locally nilpotent (locally supersoluble).

"Proof of the Theorem".

Assume that the Theorem is false.
Consider a minimal counterexample G with respect to the Chernikov parameter Θ.

1. Then $G = AB = AC = BC$ is a soluble Chernikov group with minimal $\Theta(G)$, A and B are locally nilpotent, C is an $\mathfrak{F}$-group, but G is not an $\mathfrak{F}$-group.
2. Every such trifactorized Chernikov group with $\Theta(X) < \Theta(G)$ is an $\mathfrak{F}$-group.
3. The maximal q-subgroups of G are conjugate for all primes q.
4. $J(G) = J(A)J(B)$.
5. $J(G)$ is an abelian p-group for some prime p, $G/J(G)$ is a finite $\mathfrak{F}$-group.
6. The maximal p'-subgroups of G are finite.

7. There exists a maximal p'-subgroup H of G such that $H = A'_p B'_p$ where A'_p is the maximal p'-subgroup of A and B'_p is the maximal p'-subgroup of B.
8. $G = PH$ where P is a maximal p-subgroup of G
9. The maximal normal p'-subgroup $O'_p(G)$ of G is trivial.
10. The subgroups A'_p and B'_p are both nontrivial and $(A'_p)^x$ and $(B'_p)^y$ centralize each other for all $x, y \in G$.
11. $N_1 = C_G((A'_p)^G)$ and $N_2 = C_G((B'_p)^G)$ are normal subgroups of G with $\mathfrak{F}$-factor groups, since $\Theta(G/N_i) < \Theta(G)$.
12. If $T = N_1 \cap N_2$, then G/T is an $\mathfrak{F}$-group.
13. T is **hypercentrally embedded** in G, i.e. there exists a series of normal subgroups of G contained in T, so that G acts trivially on the factors.
14. G is an $\mathfrak{F}$-group.
15. This contradiction proves the theorem.

Corollary.

Let $G = AB = AC = BC$ a trifactorized locally finite group, whose p-subgroups satisfy Min for every prime p (they are then locally nilpotent).

Let the subgroups A and B be locally nilpotent.

If C belongs to a subgroup closed saturated formation $\mathfrak{F}$ containing all locally nilpotent groups, then G is also in $\mathfrak{F}$.

In particular, if C is locally nilpotent (locally supersoluble), then G is locally nilpotent (locally supersoluble).

Products of Chernikov groups.

Problem.

Is every group $G = AB$ which is the product of two Chernikov subgroups A and B always a Chernikov group?

Theorem (B. A., L. Kazarin, Arch. Math, to appear).

Let the group $G = AB$ be the product of two Chernikov subgroups A and B, **which both have abelian subgroups** A_0 **and** B_0 **respectively with index at most** 2.

Let further one of the two subgroups, A say, be of dihedral type, i.e. A contains an involution τ which inverts every element of A_0.

Then G **is also a Chernikov group**.

Problem.

Let $G = AB = AC = BC$ be trifactorized.

Is G a Chernikov group, if A, B and C are Chernikov groups?
`(Problem 13.27 of Kourovka Note Book)`

Conjecture. If A, B, C have Min, then G in general does not have Min.

Question. Is G a Chernikov group, if A, B, C have are Chernikov groups with $A/J(A)$, $B/J(B)$ and $C/J(C)$ of index at most 2?