

# Rings with pathological matrix rings

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25th April 2009

## Definition

*matrix type of a ring  $R$ :*

$$\text{mt}(R) := \{(n, m) \mid M_n(R) \cong M_m(R)\}.$$

## Theorem (BG)

*$k$  field:*

$$\{\text{mt}(R) \mid R \text{ ultramatricial } k\text{-algebra}\} = \{\equiv \mid nk \equiv mk \implies n \equiv m\}.$$

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$$K_0(R) := (\langle \{P \leq_{\oplus} R^n \mid n \in \mathbb{N}\}, \oplus, \leq, R),$$
$$\{x \in K_0(R) \mid x \geq 0\} := \{P \leq_{\oplus} R^n \mid n \in \mathbb{N}\}.$$

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# Ultramatricial algebras and dimension groups

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ultramatricial algebra:

$$R = \bigcup_{n=0}^{\infty} R_n, \quad \text{where } R_n = \prod_{i=0}^{k(n)} M_{m(i,n)}(k)$$

dimension group:

$$(D, \leq, u) = \bigcup_{n=0}^{\infty} (\mathbb{Z}, \leq)^{k(n)} \quad \text{with } D \leq \mathbb{N}u.$$

## Theorem (Elliott)

$K_0$ :  $\{\text{ultramatricial algebras}\} / \cong = \{\text{dimension groups}\} / \cong$ .

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# Characterization of matrix types of ultramatricial algebras

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$$\begin{aligned}M_n(R) \cong M_m(R) &\iff K_0(M_n(R)) \cong K_0(M_m(R)) \\ &\iff (K_0(R), \leq, nR) \cong (K_0(R), \leq, mR).\end{aligned}$$

- $(D, \leq, u)$  dimension group:

$$\exists! H \leq \mathbb{Q}_+^\times : (D, \leq, nu) \cong (D, \leq, mu) \iff \frac{n}{m} \in H$$

## Theorem (BG)

- $H \leq \mathbb{Q}_+^\times$
- $\implies \exists (D, \leq, u)$  dimension group:
- $\text{Aut}(D, \leq) \cong H$
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## Theorem (A.L.S. Corner)

- $R$  ring,  $(R, +)$  free
- ${}_R B$  free module
- $\{w_b : b \in B \setminus \{0\}\} \subseteq \widehat{\mathbb{Z}}$  algebraically independent

$\implies G := \langle B, Rbw_b : b \in B \setminus \{0\} \rangle_* \subseteq \widehat{B}$

- $\text{End } G = R.$

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- $R = \mathbb{Z}H$ ,  $\text{rank } B = \aleph_0.$
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  - Hard part:  $\leq$  such that  $(D, \leq, u)$  dimension group
- $\implies \text{Aut}(D, \leq) = H.$

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# An easy dimension group

## Order on free base

- $t_n \in \mathbb{Z}^+$
- $B := \langle y \in {}^{<\omega}2 \mid y = y_0 + t_n \cdot y_1 \rangle$  free
- $B_n := \prod_{y \in {}^{n}2} (\mathbb{Z}y, \leq) \subseteq B$
- $(1, t_n): B_n \rightarrow B_n \times B_n \cong B_{n+1}$  reflects order

## Dimension group $(\mathbb{Q}u \oplus B, \leq, u)$

$$\mathbb{Q}u \oplus B_n = \mathbb{Q}(u - \sum_{y \in {}^{n}2} y) \times (B_n, \leq)$$

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## Remark

$D$  dimension group:

$$\text{rank } D < \infty \implies (D, \leq) = \prod (D_i, \leq) \quad D_i \subseteq \mathbb{Q}.$$

Division in dimension groups

$$\bullet A = \langle x_0, \frac{x_1}{p} \rangle \subseteq (\mathbb{Q}R_0, \leq) \times (\mathbb{Q}R_1, \leq)$$

$$\bullet y = y_0 + px + y_1$$

$$\langle x_0, \frac{x_1}{p} \rangle + \langle y_0, y_1 \rangle = \langle x_0 + y_0, \frac{x_1}{p} + y_1 \rangle$$

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- $A := \langle c, y, \frac{c-y}{p} \rangle \subseteq (\mathbb{Q}Rc, \leq) \times (\mathbb{Q}Ry, \leq)$
- $y = y_0 + pk \cdot y_1$
- $A \oplus Ry_1 = \left( R \left( \frac{c - y_0}{p} \right), \leq \right) \times (Ry_0, \leq) \times (Ry_1, \leq)$   
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# Order and structure of main part

- $R := \mathbb{Z}H = \bigoplus_{h \in H} (\mathbb{Z}x_h, \leq)$
- $t_n \in \mathbb{Z}^+$  such that  $\forall m \forall n \gg m: m \mid t_n$
- $B := \langle y \in {}^{<\omega}2 \mid y = y_0 + t_n \cdot y_1 \rangle_R$  free
- $B_n := \prod_{y \in {}^{n_2}} (Ry, \leq) \subseteq B$

- $\mathbb{Z}$ -adic integers:  $w_b^{(n)} \in \widehat{\mathbb{Z}}$

- $w_b^{(0)} = w_b$

- $w_b^{(n)} - t_n w_b^{(n+1)} \in \mathbb{Z}, \quad \forall m \forall n \gg m: m \mid t_1 \cdots t_n$

- $G := \langle B, Rbw_b^{(n)} : b \in B \setminus \{0\}, n \in \mathbb{N} \rangle$
- Partially-ordered subgroups: ( $n$  fixed)

$$(B, \leq) \times \bigoplus_{b \in B} \underbrace{(R(bw_b^{(n)}), \leq)}_{\text{isomorphic to } (B, \leq)}$$

- $c_b^{(n)} - t_n c_b^{(n+1)} = \sum_{y \in {}^{n_2}} ? \cdot y_0$

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  - $w_b^{(n)} - t_n w_b^{(n+1)} \in \mathbb{Z}, \quad \forall m \forall n \gg m: m \mid t_1 \cdots t_n$
- $G := \langle B, Rbw_b^{(n)} : b \in B \setminus \{0\}, n \in \mathbb{N} \rangle$
- Partially-ordered subgroups: ( $n$  fixed)

$$(B, \leq) \times \bigoplus_{b \in B} \underbrace{(R(bw_b^{(n)} + d(b, n)), \leq)}_{c_b^{(n)}} \quad d(b, n) \in B.$$

- $c_b^{(n)} - t_n c_b^{(n+1)} = \sum_{y \in {}^{n_2}} ? \cdot y_0$

# Order and structure of main part

- $R := \mathbb{Z}H = \bigoplus_{h \in H} (\mathbb{Z}x_h, \leq)$
- $t_n \in \mathbb{Z}^+$  such that  $\forall m \forall n \gg m: m \mid t_n$
- $B := \langle y \in {}^{<\omega}2 \mid y = y_0 + t_n \cdot y_1 \rangle_R$  free
- $B_n := \prod_{y \in {}^{n_2}} (Ry, \leq) \subseteq B$
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# Adding an order unit

## An order of a free group

- $s_n \in \mathbb{Z}^+$  such that  $\forall m \forall n \gg m: m \mid s_n$
- $k_n := s_1 \dots s_n$
- $B := \langle y \in {}^{<\omega}2 \mid y = y_0 + y_1 \rangle$  free Abelian group.
- $B_n := \prod_{y \in {}^{n_2}} (\mathbb{Z}y, \leq) \subseteq B$

## Becomes a dimension group

- $\mathbb{Q}u \oplus B$ , where  $u$  is an order unit.
- $\mathbb{Z}u \oplus B_n = \mathbb{Z}(u - \underbrace{\sum_{y \in {}^{n_2}} y}_{v_n}), \leq) \times (B_n, \leq)$
- $v_n - s_n v_{n+1} = \sum_{y \in {}^{n_2}} y_0$

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## Becomes a dimension group

- $\mathbb{Q}u \oplus B$ , where  $u$  is an order unit.
- $\mathbb{Z} \frac{u}{k_n} \oplus B_n = \mathbb{Z} \left( \underbrace{\frac{u}{k_n} - k_n \sum_{y \in {}^n 2} y}_{v_n}, \leq \right) \times (B_n, \leq)$
- $v_n - s_n v_{n+1} = \sum_{y \in {}^n 2} ? \cdot y_0$

# Putting the order together

$$D := \mathbb{Q}u \oplus G$$

## Partially-ordered subgroups

- $G_n = B_n \times \prod_{i=1}^n Rc_{b_i}^{(n)} \subseteq G$
- $\mathbb{Z}v_n \times B_n$  ,

$$v_n = \frac{u}{k_n} - \left( k_n \sum_{y \in \mathbb{N}^2} y + k_n \sum_{i=1}^n c_{b_i}^{(n)} \right)$$

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$$v_{n,F} = \frac{u}{k_n} - \sum_{h \in F} h^{-1} x_h \left( k_n \sum_{y \in \mathbb{N}^{n_2}} y + l_n \sum_{i=1}^n c_{b_i}^{(n)} \right)$$

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