Flat Model Structures for Nonunital Rings

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Outline

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A: nonunital k-algebra (or an associative ring without 1).

Definition

"A" satisfies excision in Bar, cyclic and Hochschild (or in algebraic K-theory) if the corresponding homology groups (or K-groups) does not depend upon the extension of A into a unital k-algebra R.

H-unital algebras

M. Wodzicki, *Excision in cyclic homology and in rational algebraic K-theory*, Ann. of Math. **129** (1989), 591-639.

$\tilde{A} = k \ltimes A$.

A is H-unital if $\mathrm{Tor}^{n}_{\tilde{A}}(k, A) = 0$, $\forall n \geq 0$.

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Theorem (Wodzicki)

FAE:

- *A is H-unital.*
- *A satisfies excision in cyclic homology.*
- *A satisfies excision in Hochschild homology.*
- *A satisfies excision in Bar homology.*

A. A. Suslin and M. Wodzicki, *Excision in algebraic K-theory*, Ann. of Math. **136** (1992), 51-122.

Theorem (Suslin-Wokzicki)

Let A be $\mathbb{Q}$-algebra.
A satisfies excision in K-Theory $\Leftrightarrow$ A is H-unital.

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$\Rightarrow$ Approximations to the different homology theories by using quasi-free resolutions.

D. Quillen, K_0 *for nonunital rings and Morita invariance*, J. Reine Angew. Math. **472**(1996), 197-217.

$\star$ Quillen defines a variant of $K_0 A$ for rings without 1, which is Morita invariant in very general terms.

This alternate definition is made *out* of the category of A-modules (by considering projective resolutions of $\tilde{A}$-modules.)

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Our Goal: Define the suitable framework to develop a version of Morita invariant relative homological algebra for nonunital algebras.

Definition (Quillen)

A nonunital ring is h-unital if $\mathrm{Tor}^{n}_{\tilde{A}}(\mathbb{Z}, A) = 0$, $\forall n \geq 0$.

Definition (Suslin-Wodzicki)

Let $\tilde{A} = \mathbb{Z} \ltimes A$. *A left A-module M is h-unitary if* $A \otimes_{\tilde{A}} M \cong M$ *and* $\mathrm{Tor}^{n}_{\tilde{A}}(A, M) = 0, \forall n \geq 1$ *or, equivalently, if* $\mathrm{Tor}^{n}_{\tilde{A}}(\mathbb{Z}, M) = 0$, $\forall n \geq 0$.

S. Estrada and P.A. Guil-Asensio, *Quillen monoidal model structures for nonunital rings*, submitted.

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  - Every A-module M admits a right minimal approximation by an h-unitary M_h.
  - Define the homology of A from its minimal approximation by an h-unital A_h.

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  - Every A-module M admits a right minimal approximation by an h-unitary M_h.
  - Define the homology of A from its minimal approximation by an h-unital A_h.
- The category $\mathbb{C}(\mathfrak{M}(A))$ admits a Flat Model structure (in the sense of Quillen) $\Rightarrow \mathbb{D}(\mathfrak{M}(A))$.

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Problem: $\mathfrak{M}(A)$ does not have in general enough projectives!.

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Problem: these resolutions are not unique up to homotopy, so they are not suitable to define torsion functors.

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Quillen Model Structure

D. Quillen, *"Homotopical Algebra"*, Lecture Notes in Mathematics, vol. 43, Springer-Verlag, Berlin/New York, 1967.

Idea: To impose a Flat Model Structure compatible with the tensor product in $\mathbb{C}(\mathfrak{M}(A))$.

Advantages:

- The derived category $\mathbb{D}(\mathfrak{M}(A))$ does not depend of the extension of A by R (with 1).
- It allows to define homology and cohomology functors in $\mathfrak{M}(A)$.

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$\mathfrak{M}(A)$ is a λ-accesible category.

$\mathcal{F}$: the class of all flat and h-unitary modules.

$\mathcal{C} = \{C \in \mathfrak{M}(A) : \mathrm{Ext}^1_{\mathfrak{M}(A)}(F, C) = 0, \ \forall F \in \mathcal{F}\}$

Outline

$\mathbb{C}(\mathfrak{M}(A))$: category of unbounded complexes of h-unitary modules.

$$M = \cdots \to M^{i-1} \xrightarrow{\delta^{i-1}} M^i \xrightarrow{\delta^i} M^{i+1} \to \cdots$$

M^i is h-unitary, $\forall i \in \mathbb{Z}$.
$Z_i M$ the i'th cyclic module $\ker(\delta^i)$
$B_i M$, el i'th boundary module $\mathrm{Im}\,(\delta^{i-1})$.

1. $C \in \mathbb{C}(\mathfrak{M}(A))$ is $\mathcal{F}$-cotorsion if C is exact and $Z_n C \in \mathcal{C}$, for all $n \in \mathbb{Z}$. $\widetilde{\mathcal{C}}$: the class of all $\mathcal{F}$-cotorsion complexes.
2. $F \in \mathbb{C}(\mathfrak{M}(A))$ is h-unitary dg-flat if F^n is h-unitary flat for all $n \in \mathbb{Z}$ and, for each $\mathcal{F}$-cotorsion complex C, $Hom(F, C)$ is an exact complex in $\mathbb{Z}$-Mod. $dg\widetilde{\mathcal{F}}$: the class of h-unitary dg-flat complexes.

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3. $X \in \mathbb{C}(\mathfrak{M}(A))$ is h-unitary flat if X is exact and $Z_n X \in \mathcal{F}$, $\forall n \in \mathbb{Z}$. $\widetilde{\mathcal{F}}$: the class of all flat h-unitary complexes.
4. $Y \in \mathbb{C}(\mathfrak{M}(A))$ is dg-cotorsion if $Y_n \in \mathcal{C}$ and $Hom(X, Y)$ is exact, for each h-unitary flat complex para X. $dg\widetilde{\mathcal{C}}$: the class of dg-cotorsion complexes.

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Theorem (Hovey)

$\mathcal{E}$: the class of all exact complexes in $\mathfrak{M}(A)$.

If $(dg\widetilde{\mathcal{F}}, dg\widetilde{\mathcal{C}} \cap \mathcal{E})$ and $(dg\widetilde{\mathcal{F}} \cap \mathcal{E}, dg\widetilde{\mathcal{C}})$ are complete cotorsion pairs

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1. The pairs $(\widetilde{\mathcal{F}}, dg\widetilde{\mathcal{C}})$ y $(dg\widetilde{\mathcal{F}}, \widetilde{\mathcal{C}})$ are cotorsion pairs.
2. The pairs $(\widetilde{\mathcal{F}}, dg\widetilde{\mathcal{C}})$ y $(dg\widetilde{\mathcal{F}}, \widetilde{\mathcal{C}})$ are complete.
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Theorem

The previous Model Structure in $\mathbb{C}(\mathfrak{M}(A))$ is compatible with the usual tensor product of complexes.

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Definition

Given M and N h-unitary modules. We define $\mathrm{Ext}_A^n(M, N)$ as

$$\mathrm{Hom}_{\mathbb{D}(\mathfrak{M}(A))}(S(M), S^n(N)) \cong \mathrm{Hom}_{\mathbb{C}(\mathfrak{M}(A))}(Q, P)/\sim$$

where Q is a cofibrant replacement of $S(M)$ and P is a fibrant replacement of $S^n(N)$ ($\sim$ denotes the equivalence class given by the quasi-isomorphisms).

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