Endomorphism rings with two (or finitely many) maximal ideals

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Dedicated to Rüdiger

Krull-Schmidt-Azumaya Theorem in the case two

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Two joint papers with P. Příhoda (Charles University in Prague).

Recall that if R and S are rings, a ring homomorphism $\varphi\colon R \to S$ is *local* if, for every $r \in R$, $\varphi(r)$ invertible in S implies r invertible in R.

We say that a ring R *has type* n if the factor ring $R/J(R)$ is a direct product of n division rings.

Proposition 1 *The following conditions are equivalent for a positive integer n and a ring R with Jacobson radical $J(R)$.*

(i) *n is the smallest integer m such that there exists a local homomorphism of the ring R into a direct product of m division rings.*

(ii) *R has exactly n distinct maximal right ideals, and they are all two-sided ideals in R.*

(iii) *R has exactly n distinct maximal left ideals, and they are all two-sided ideals in R.*

(iv) *The ring R has type n (i.e., $R/J(R)$ is a direct product of n division rings).*

Moreover, if these equivalent conditions hold and, for every $i = 1, \dots, n$, $\varphi_i \colon R \to D_i$ *is a ring morphism of* R *into a division ring* D_i *with*

$$\varphi_1 \times \cdots \times \varphi_n \colon R \to D_1 \times \cdots \times D_n$$

a local morphism, then $\ker \varphi_1, \dots, \ker \varphi_n$ *are exactly the* n *distinct maximal right ideals and maximal left ideals of* R*.*

A ring R has type 1 if and only if it is a local ring, if and only if there is a local morphism of R into a division ring.

First example

We say that two modules A_R and B_R *have the same upper part*, and write $[A_R]_u = [B_R]_u$, if there exist a homomorphism $\varphi\colon E(A_R) \to E(B_R)$ and a homomorphism $\psi\colon E(B_R) \to E(A_R)$ such that $\varphi^{-1}(B_R) = A_R$ and $\psi^{-1}(A_R) = B_R$.

We say that A_R and B_R *belong to the same monogeny class* ($[A_R]_m = [B_R]_m$) if there exist a monomorphism $A_R \to B_R$ and a monomorphism $B_R \to A_R$.

Let E_1, E_2, E'_1, E'_2 be indecomposable injective right modules over an arbitrary ring R, and let $\varphi\colon E_1 \to E_2, \varphi'\colon E'_1 \to E'_2$ be two non-injective morphisms. Any morphism $f\colon \ker\varphi \to \ker\varphi'$ extends to the injective resolutions $0 \to \ker\varphi \to E_1 \xrightarrow{\varphi} E_2$ and $0 \to \ker\varphi' \to E'_1 \xrightarrow{\varphi'} E'_2$ of $\ker\varphi$ and $\ker\varphi'$. Thus we have a commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \to & \ker\varphi & \to & E_1 & \xrightarrow{\varphi} & E_2 \\ & & \downarrow f & & \downarrow f_1 & & \downarrow f_2 \\ 0 & \to & \ker\varphi' & \to & E'_1 & \xrightarrow{\varphi'} & E'_2. \end{array}$$

Notice that f_1 and f_2 are not uniquely determined by f.

Theorem 2 (F.-Herbera) *Let E_1 and E_2 be two indecomposable injective right modules over an arbitrary ring R, and let $\varphi\colon E_1 \to E_2$ be a non-zero non-injective morphism. The mapping*

$$\begin{array}{l} \mathrm{End}_R(\ker\varphi) \to \mathrm{End}_R(E_1)/J(\mathrm{End}_R(E_1))\times \\ \qquad\qquad\qquad \times\, \mathrm{End}_R(E_2)/J(\mathrm{End}_R(E_2)) \\ f \mapsto (f_1 + J(\mathrm{End}_R(E_1)), f_2 + J(\mathrm{End}_R(E_2))) \end{array}$$

is a well defined local morphism. Hence $\mathrm{End}_R(\ker\varphi)$ *has type* ≤ 2*.*

Theorem 3 (Weak Krull-Schmidt Theorem; F.- Ecevit-Koşan-Özdin) *Let* $\varphi_i \colon E_{i,1} \to E_{i,2}$ $(i = 1, 2, \ldots, n)$ *and* $\varphi'_j \colon E'_{j,1} \to E'_{j,2}$ $(i = 1, 2, \ldots, t)$ *be non-injective morphisms between indecomposable injective modules* $E_{i,1}, E_{i,2}, E'_{j,1}, E'_{j,2}$ *over an arbitrary ring* R. *Then* $\oplus_{i=1}^{n} \ker \varphi_i \cong \oplus_{j=1}^{t} \ker \varphi'_j$ *if and only if* $n = t$ *and there exist two permutations* σ, τ *of* $\{1, 2, \ldots, n\}$ *such that* $[\ker \varphi_i]_m = [\ker \varphi'_{\sigma(i)}]_m$ *and* $[\ker \varphi_i]_u = [\ker \varphi'_{\tau(i)}]_u$ *for every* $i = 1, 2, \ldots, n$.

Weak Krull-Schmidt Theorem=isomorphism classes are defined by two permutations.

Cyclically presented modules

Joint paper with Babak Amini and Afshin Amini (Shiraz University, Iran), to appear in J. Algebra.

A right module over a ring R is said to be *cyclically presented* if it is isomorphic to R/aR for some $a \in R$.

If R/aR and R/bR are cyclically presented R-modules, R local , we say that R/aR and R/bR *have the same lower part*, and write $[R/aR]_l = [R/bR]_l$, if $\exists u, v \in U(R), r, s \in R$ with $au = rb$ and $bv = sa$.

R/aR non-zero cyclically presented $\Longrightarrow$ $\mathrm{End}_R(R/aR) \cong E/aR$, where $E := \{ r \in R \mid ra \in aR \}$ is the *idealizer* of aR.

Theorem 4 *Let a be a non-zero non-invertible element of a local ring R, let E be the idealizer of aR, and let E/aR be the endomorphism ring of the cyclically presented right R-module R/aR. Let $I := \{ r \in R \mid ra \in aJ(R) \}$ and $K := J(R) \cap E$. Then one of the following two conditions hold:*

(a) *Either E is a local ring, or*

(b) $E/J(E) \cong E/I \times E/K$*, where E/I and E/K are division rings.*

Theorem 5

(Weak Krull-Schmidt Theorem)

Let $a_1, \ldots, a_n, b_1, \ldots, b_t$ *be non-invertible elements of a local ring* R. *Then*

$$R/a_1R \oplus \cdots \oplus R/a_nR \cong R/b_1R \oplus \cdots \oplus R/b_tR$$

as right R*-modules if and only if* $n = t$ *and there are two permutations* σ, τ *of* $\{1, 2, \ldots, n\}$ *such that* $[R/a_iR]_l = [R/b_{\sigma(i)}R]_l$ *and* $[R/a_iR]_e = [R/b_{\tau(i)}R]_e$ *for every* $i = 1, 2, \ldots, n$.

Corollary 6 *Let* $a_1, \ldots, a_n, b_1, \ldots, b_n \in R$, R *local. Then* $\mathrm{diag}(a_1, \ldots, a_n) \sim \mathrm{diag}(b_1, \ldots, b_n)$ $\iff$ $\exists \sigma, \tau$ *with* $[R/a_iR]_l = [R/b_{\sigma(i)}R]_l$ *and* $[R/a_iR]_e = [R/b_{\tau(i)}R]_e$ *for every* i.

Further examples with exactly the same behaviour:

couniformly presented modules (F.-Girardi);

modules with Goldie dimension 1 and dual Goldie dimension 1 (F.).

We will say that a right module M_R over a ring R *has type* n if its endomorphism ring $\mathrm{End}(M_R)$ is a ring of type n.

General theory for modules of type n?

$\mathcal{C} =$ a full subcategory of Mod-R.

M a non-zero right R-module, P a two-sided ideal of $\mathrm{End}_R(M)$. Let $\mathcal{P}$ be the ideal of the category $\mathcal{C}$ (the *ideal of* $\mathcal{C}$ *associated to* P) defined as follows: a morphism $f\colon X \to Y$ is in $\mathcal{P}(X,Y)$ if and only if $\beta f \alpha \in P$ for every $\alpha\colon M \to X$ and every $\beta\colon Y \to M$. If M is an object of $\mathcal{C}$, then $\mathcal{P}$ is the greatest among the ideals $\mathcal{P}'$ of $\mathcal{C}$ with $\mathcal{P}'(M,M) \subseteq P$, and in this case, as is easily seen, $\mathcal{P}(M,M) = P$.

For every module M of type n, set $V(M_R) =$ "the set whose elements are the n ideals $\mathcal{P}_1, \dots, \mathcal{P}_n$ of the category $\mathcal{C}$ associated to the n maximal ideals $P_1, \dots, P_n$ of $\mathrm{End}_R(M)$". (The set $V(M_R)$ has cardinality exactly n.)

Theorem 7 *Let $\mathcal{C}$ be a full subcategory of* Mod-R *and $M, N \in \mathrm{Ob}(\mathcal{C})$ be R-modules of finite type. Then $M \cong N$ if and only if $V(M) = V(N)$.*

FT-R = all right R-modules of finite type.

SFT-R = all right R-modules that are direct sums of finitely many right R-modules of finite type.

add($\mathcal{C}$) = closure of $\mathcal{C}$ with respect to direct summands, $\mathcal{K}$ = ideal of add($\mathcal{C}$) associated to P.

Proposition 8 *Let M be a right R-module of finite type, let P be a maximal ideal of* $\mathrm{End}_R(M)$*, let $\mathcal{P}$ denote the ideal of* SFT-R *associated to P and $\mathcal{K}$ the ideal of* add(SFT-R) *associated to P. Then* SFT-$R/\mathcal{P} \cong$ add(SFT-R)$/\mathcal{K} \cong$ mod-$\mathrm{End}_R(M)/P$*.*

Let $\mathcal{F}$ be the class of all canonical functors $F\colon$ add(SFT-R) $\to$ add(SFT-R)$/\mathcal{P}$, where $\mathcal{P} \in V(M_R)$ for some $M_R \in$ Ob(FT-R).

For every $F \in \mathcal{F}$, we can define $\dim_F(M)$ as the dimension of the vector space over End$(M)/P$ corresponding to $F(M)$.

Corollary 9 *Let M, N be objects of* add(SFT-R). *Then $M \cong N$ if and only if* $\dim_F(M) = \dim_F(N)$ *for every $F \in \mathcal{F}$.*

The Krull-Schmidt-Azumaya Theorem in the case 2.

Module of type 1 = module whose endomorphism is local.

The Krull-Schmidt-Azumaya Theorem: if $M_1, \dots, M_m, N_1, \dots, N_n$ are R-modules with local endomorphism ring and $M_1 \oplus \dots \oplus M_m \cong N_1 \oplus \dots \oplus N_n$, then $n = m$ and there is a permutation σ of $\{1, \dots, n\}$ such that $M_i \cong N_{\sigma(i)}$ for every $i = 1, \dots, n$.

Equivalently, the monoid of isomorphism classes of the modules that are finite direct sums of modules with local endomorphism ring is a free commutative monoid. A free set of generators of this monoid is given by the isomorphism classes of the modules with local endomorphism ring.

the isomorphism class of $A_R = \langle A_R \rangle$

$:= \{\, X_R \in \text{Mod-}R \mid X_R \cong A_R \text{ in Mod-}R \,\}$

Fix a class $\mathcal{C} \subseteq$ Mod-R. Assume that $\mathcal{C}$ is closed under isomorphism, direct summands and finite direct sums.

Set

$$V(\mathcal{C}) := \{ \langle A_R \rangle \mid A_R \in \mathcal{C} \}.$$

($V(\mathcal{C})$ is a class, not a set in general).

Define

$$\langle A_R \rangle + \langle B_R \rangle := \langle A_R \oplus B_R \rangle$$

for every $A_R, B_R \in \mathcal{C}$. Then $V(\mathcal{C})$ becomes an additive commutative monoid.

Example: (Krull-Schmidt-Azumaya Theorem)

$\mathcal{C} = \{ M_R \mid M_R$ is the direct sum of finitely many modules of type $1 \}$

$V(\mathcal{C})$ is a free commutative monoid:

$$V(\mathcal{C}) \cong \mathbf{N}^{(X)}.$$

Type ≤ 2

R a fixed ring.

$\mathcal{T} = \{$ indecomposable right R-modules of type $\leq 2 \}$. Does some weak form of the Krull-Schmidt-Azumaya Theorem holds?

The situation is described by a *graph* G *associated to* R.

Vertices: the ideals $\mathcal{P}$, where $\mathcal{P}$ is the ideal of the full subcategory $\mathcal{T}$ associated to a maximal ideal P of $\mathrm{End}(M_R)$ for some $M_R \in \mathcal{T}$. $\quad \bigcup_{M_R \in \mathcal{T}} V(M_R)$

Edges: the isomorphism classes $\langle M \rangle := \{ Y \in \text{Mod-}R \mid Y \cong M \text{ in Mod-}R \}$ where M_R is any R-module of type 2. (For every such M_R, the endomorphism ring $\mathrm{End}(M_R)$ has exactly two maximal ideals P_1, P_2, and the edge $\langle M \rangle$ connects the vertices $\mathcal{P}_1$ and $\mathcal{P}_2$.)

Theorem 10 *For every ring R, the connected components of the graph G are either complete graphs K_α ($\alpha \geq 1$ a cardinal) or complete bipartite graphs $K_{\alpha,\beta}$ ($\alpha \geq \beta \geq 1$ cardinals).*

We can associate a commutative monoid $V(G)$ to any graph $G=(V,E)$. Given a graph $G=(V,E)$, where the elements of E are subsets of V of cardinality 2, consider the free commutative monoid $\mathbb{N}_0^{(V)}$ having as free set of generators the set of all $\delta_v\colon V\to\mathbb{N}_0$, $v\in V$, with $\delta_v(v)=1$ and $\delta_v(w)=0$ for every $w\in V$, $w\neq v$. If $\ell=\{v,w\}\in E$ is an edge of G, define $\delta_\ell:=\delta_v+\delta_w\in\mathbb{N}_0^{(V)}$. Let $V(G)$ be the submonoid of $\mathbb{N}_0^{(V)}$ generated by (1) all the elements $\delta_\ell\in\mathbb{N}_0^{(V)}$, where ℓ ranges in E, and

(2) all the elements δ_v, where v ranges in the isolated vertices of G.

R a ring

$\mathcal{C} = \{ M_R \mid M_R$ is the direct sum of finitely many modules of type $\leq 2 \}$.

Every module in $\mathcal{C}$ has a decomposition, unique up to isomorphism, indexed in the set of all connected components of G (because if C_λ ($\lambda \in \Lambda$) are the connected components of G, then

$$V(\mathcal{C}) = V(G) = \oplus_{\lambda \in \Lambda} V(C_\lambda),$$

so that every element of $V(\mathcal{C})$ is the sum of elements in the $V(C_\lambda)$'s in a unique way.)

Proposition 11 *Let* $M_1, \ldots, M_m, N_1, \ldots, N_n$ *be right* R*-modules of type* 2*, all in the same connected component of* G*. Assume that this connected component is a complete graph. Let* P_1, P_2 *be the two maximal ideals of* M_1*, ...,* P_{2m-1}, P_{2m} *the two maximal ideals of* M_m*,* Q_1, Q_2 *be the two maximal ideals of* N_1*, ...,* Q_{2n-1}, Q_{2n} *the two maximal ideals of* N_n*. Let* $\mathcal{P}_i$ $(i = 1, \ldots, m)$*,* $\mathcal{Q}_j$ $(j = 1, \ldots, n)$ *be the corresponding associated ideals in* Mod-R*. Then* $M_1 \oplus \cdots \oplus M_m \cong N_1 \oplus \cdots \oplus N_n \Leftrightarrow m = n$ *and* $\exists$ *a permutation* σ *of* $\{1, 2, \ldots, 2n\}$ *such that* $\mathcal{P}_i = \mathcal{Q}_{\sigma(i)}$ *for every* $i = 1, 2, \ldots, 2n$*.*

In particular, under the hypotheses of Proposition 11, the module $M_1 \oplus \cdots \oplus M_m$ has exactly $\frac{(2m)!}{2^m \cdot m!}$ non-isomorphic direct-sum decompositions into direct sums of modules of type 2. Similarly:

Proposition 12 *$C = (V_C, E_C)$ a connected component of G, C a bipartite complete graph, $V_C = X_C \dot\cup Y_C$ the corresponding bipartition. Let $M_1, \dots, M_m$, $N_1, \dots, N_n$ be right R-modules of type 2 in the component C. It is possible to label the maximal ideals of* $\mathrm{End}_R(M_i)$ *and* $\mathrm{End}_R(N_j)$ *in such a way that the associated ideals $\mathcal{P}_1, \dots, \mathcal{P}_m, \mathcal{P}'_1, \dots, \mathcal{P}'_n$ in X corresponds to the maximal ideals $P_1, \dots, P_m, P'_1, \dots, P'_n$ of* $\mathrm{End}(M_1), \dots, \mathrm{End}(M_m), \mathrm{End}(N_1), \dots, \mathrm{End}(N_n)$ *respectively, and the associated ideals $\mathcal{Q}_1, \dots, \mathcal{Q}_m, \mathcal{Q}'_1, \dots, \mathcal{Q}'_n$ in Y corresponds to the maximal ideals $Q_1, \dots, Q_m, Q'_1, \dots, Q'_n$ of*

$\mathrm{End}(M_1), \ldots, \mathrm{End}(M_m), \mathrm{End}(N_1), \ldots, \mathrm{End}(N_n)$ *respectively. Then* $M_1 \oplus \cdots \oplus M_m \cong N_1 \oplus \cdots \oplus N_n$ *if and only if* $m = n$ *and there exist two permutations* σ, τ *of* $\{1, \ldots, n\}$ *such that* $\mathcal{P}_i = \mathcal{P}'_{\sigma(i)}$ *and* $\mathcal{Q}_i = \mathcal{Q}_{\tau(i)}$ *for every* $i = 1, \ldots, n$*,*

In particular, under the hypotheses of Proposition 12, the module

$$M_1 \oplus \cdots \oplus M_m$$

has exactly $n!$ direct-sum decompositions into a direct sum of modules of type 2.