Characteristic and Fully Invariant Subgroups of Abelian p-Groups.

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To Rüdiger, a truly great friend

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Basic Notions I

Throughout the discussion all groups (except automorphism groups) will be Abelian p-groups.

- A subgroup H of a group G is said to be fully invariant in G if $H\varphi \leq H$ for all endomorphisms φ of G.

- A subgroup H of a group G is said to be characteristic in G if $H\phi \leq H$ for all automorphisms ϕ of G.

Clearly, fully invariant subgroups are characteristic but the converse fails even for finite groups. A useful example of this failure is furnished by the group $G = \langle a \rangle \oplus \langle b \rangle$, where $o(a) = 2,\ o(b) = 8$ and H is the subgroup $\{0, a + 2b, a - 2b, 4b\}$. H is certainly not fully invariant since it is not invariant under the projection onto $\langle a \rangle$, but it is characteristic.

Basic Notions II

Recall that the Ulm sequence of an element x of a p-group G is a sequence $U_G(x) = \{\alpha_i\}_{i \geq 0}$ of ordinals or symbols ∞ such that $\alpha_i = h_G(p^i x)$, the height of $p^i x$ in G.

Ordering such sequences componentwise, one is led to the notions of transitive and fully transitive groups:

G is transitive if x can be mapped to y by an automorphism of G whenever $x, y \in G$ satisfy $U_G(x) = U_G(y)$; and fully transitive if this can be accomplished by an endomorphism of G whenever $U_G(x) \leq U_G(y)$.

Basic Notions III

In his classification of fully invariant subgroups, Kaplansky made use of the notion of a *U-sequence* relative to a group G: A monotone increasing sequence $\{\alpha_i\}_{i\geq 0}$ of ordinals and symbols ∞ is said to be a U-sequence relative to G, if each $\alpha_i \leq \lambda$, the length of G and if a gap occurs between α_{n-1} and α_n, then the Ulm invariant $f_G(\alpha_{n-1}) \neq 0$.

Recall that the Ulm invariant $f_G(\alpha_{n-1})$ is defined to be the dimension of the **GF(p)**-vector space $\frac{(p^{n-1}G)[p]}{(p^n G)[p]}$.

Kaplansky's Results

If $\{\alpha_i\}$ is a U-sequence relative to G, then $G\{\alpha_i\} = \{x \in G | U_G(x) \geq \{\alpha_i\}\}$ is easily seen to be a fully invariant subgroup of G.

Classification: If G is a reduced fully transitive group, then every fully invariant subgroup of G has the form $G\{\alpha_i\}$ for some U-sequence $\{\alpha_i\}$ relative to G.

Connection: If G is reduced transitive group and $p \neq 2$, then G is fully transitive and its characteristic subgroups are fully invariant.

Extending the Classification

The difficulty in extending the classification is highlighted by the following example:

Let $H = K \oplus L = \langle a \rangle \oplus \langle b \rangle$, $o(a) = o(b) = p$, and let Φ consist of the diagonal matrices with entries from $\mathrm{End}(K)$ and $\mathrm{End}(L)$. Now apply Corner's realization theorem to obtain a group G such that $p^\omega G = H$ and $\mathrm{End}(G) \restriction H = \Phi$. (Note that G is neither transitive nor fully transitive since K, L are both fully invariant subgroups of G.)

K cannot have the form $M(\{\alpha_i\})$ for any U-sequence $\{\alpha_i\}$: $U_G(a) = (\omega, \infty, \dots)$ and so if $K = M(\{\alpha_i\})$ for some U-sequence $\{\alpha_i\}$, then $\alpha_0 \leq \omega$. But then b, must also belong to $M(\{\alpha_i\})$, implying that $b \in K$ – contradiction.

Being Less Ambitious!

A simple consequence of Kaplansky's classification is that if H is fully invariant in the reduced fully transitive group G, then $H[p] = (p^{\alpha_0}G)[p]$ where α_0 is the first term in the U-sequence corresponding to G. This leads to the notions:

• A reduced group G is said to be socle-regular if, for all fully invariant subgroups H of G, there is an ordinal α (depending on H) with $H[p] = (p^{\alpha}G)[p]$.

• A reduced group G is said to be strongly socle-regular if, for all characteristic subgroups H of G, there is an ordinal α (depending on H) with $H[p] = (p^{\alpha}G)[p]$.

Clearly a strongly socle-regular group is socle-regular.

Generalizations of Transitivity and Full Transitivity

The notions of socle-regularity and strong socle-regularity may be construed as generalizations of Kaplansky's notions of fully transitive and transitive groups:

• If G is fully transitive, then G is socle-regular.

(This follows easily from Kaplansky's classification.)

• If G is transitive, then G is strongly socle-regular.

(This follows with a little work from the fact that Ulm sequences of socle elements have a particularly simple form.)

Note that in particular, separable and totally projective groups are both socle- and strongly socle-regular.

Properties of Socle-Regular Groups

• If A is socle-regular, then so also is $p^\alpha A$ for all ordinals α.

• Suppose that $A = G \oplus H$ where H is separable, then A is socle-regular if, and only if, G is socle-regular.

• The group G is socle-regular if, and only if, the direct sum $G^{(\kappa)}$ is socle-regular for any cardinal κ.

• Let G be a group such that $G/p^\omega G$ is a direct sum of cyclic groups. Then G is socle-regular if, and only if, $p^\omega G$ is socle-regular.

• There exist groups of length $\omega + 1$ which are not socle-regular.

Properties of Strongly Socle-Regular Groups

• If A is strongly socle-regular, then so also is $p^{\alpha}A$ for all ordinals α.

• If G is a strongly socle-regular group and H is any separable group, the direct sum $A = G \oplus H$ is strongly socle-regular.

• If G is strongly socle-regular, then so also is the direct sum $A = G^{(\kappa)}$ for any cardinal κ.

• Let G be a group such that $G/p^{\omega}G$ is a direct sum of cyclic groups. Then G is strongly socle-regular if, and only if, $p^{\omega}G$ is strongly socle-regular.

• There exist groups of length $\omega + 1$ which are not strongly socle-regular.

Note how weak points 2 and 3 are compared to the corresponding results for socle-regularity!

Some Comparisons

• The class of strongly socle-regular groups is properly contained in the class of socle-regular groups; in particular, there exists a fully transitive group which is not strongly socle-regular.

The easiest example to produce is Corner's famous example of a non-transitive, fully transitive group C with first Ulm subgroup an infinite elementary group. Every subgroup of $p^\omega C$ is characteristic in this case, so C is not strongly socle-regular.

• A direct summand of a strongly socle-regular group need not be strongly socle-regular.

Let C be the same Corner group as above and set $A = C \oplus C$. Since C is fully transitive, its square A is, by a result of Files & BG, transitive and hence strongly socle-regular.

An Unexpected Connection?

Despite the fact that there are socle-regular groups which are not strongly socle-regular, there is a strong interconnection between the classes.

Theorem For a group G the following are equivalent:

(i) for all cardinals λ, $G^{(\lambda)}$ is socle-regular;

(ii) for some cardinal $\lambda > 0$, $G^{(\lambda)}$ is socle-regular;

(iii) for all cardinals $\lambda > 1$, $G^{(\lambda)}$ is strongly socle-regular;

(iv) for some cardinal $\lambda > 1$, $G^{(\lambda)}$ is strongly socle-regular.

In particular G is socle-regular $\Leftrightarrow$ $G \oplus G$ is strongly socle-regular.

Independence from Transitivity and Full Transitivity

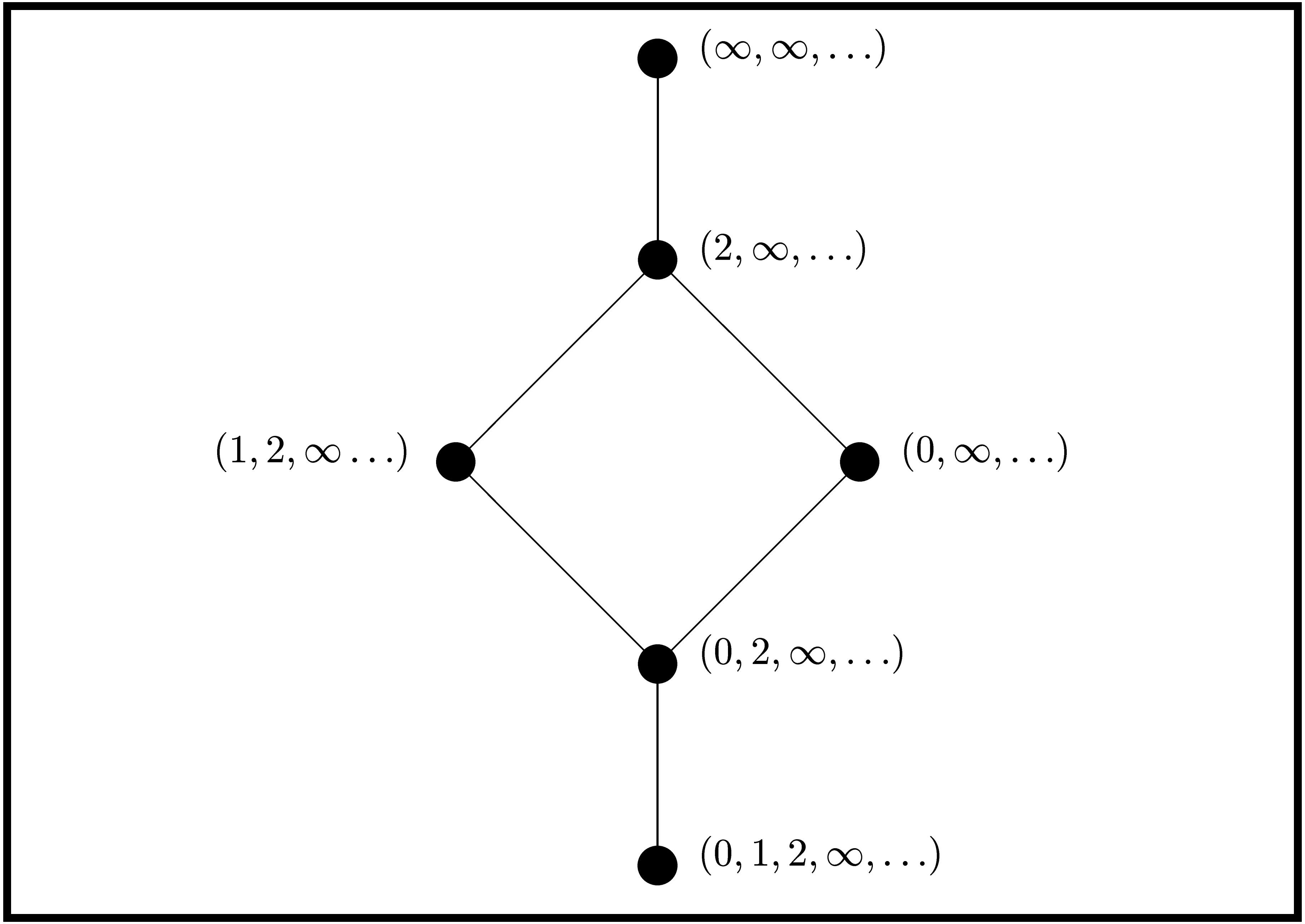
• There exists a strongly socle-regular group A which is neither transitive nor fully transitive.

Thus the class of socle-regular groups is strictly larger than the class of fully transitive groups and the class of strongly socle-regular groups is strictly larger than the class of transitive groups.

The Example

Let G be the transitive, non fully transitive 2-group constructed by Corner. The group G has the property that $2^\omega G = H$, $\mathrm{Aut}(G) \restriction 2^\omega G = \mathrm{Aut}(H)$, $\mathrm{End}(G) \restriction 2^\omega G = \Phi$, where Φ is the subring of $\mathrm{End}(H)$ generated by $\mathrm{Aut}(H)$ and the group $H = \langle a \rangle \oplus \langle b \rangle$, where a has order 2 and b has order 8. Note that H has six different associated Ulm sequences: $(\infty, \infty, \dots); (2, \infty, \dots); (0, \infty, \dots);$

$(1, 2, \infty, \dots); (0, 2, \infty, \dots); (0, 1, 2, \infty, \dots)$. It is easy to check that the only fully invariant subgroups of G contained in $2^\omega G$ are $F_1 = \{0, 4b, a - 2b, a + 2b\}$, $F_2 = \{0, a, 4b, a + 4b\}$ and $F_3 = \{0, 4b\}$. (This is essentially because it is possible to map from any vertex of the lattice, other than the vertex labelled $(0, 2, \infty, \dots)$, to any other one above it.)


(∞, ∞, . . .)
(2, ∞, . . .)
(1, 2, ∞ . . .)
(0, ∞, . . .)
(0, 2, ∞, . . .)
(0, 1, 2, ∞, . . .)

Now if F is an arbitrary fully invariant subgroup of G and $(F[2] \leq 2^\omega G)$ - this is the only important case - then $F[2]$ is one of $F_i[2]$, $i = 1, 2, 3$. However, a simple check shows that $F_1[2] = (2^{\omega+1}G)[2]$, $F_2[2] = (2^\omega G)[2]$ while $F_3[2] = (2^{\omega+2}G)[2]$. Thus the socle of each fully invariant subgroup of G is of the form $(2^\alpha G)[2]$ for some α and G is socle-regular.

Let $A = G \oplus G$ and note that A cannot be fully transitive since its direct summand G is not fully transitive. Moreover, A cannot be transitive since this would force G to be fully transitive. However, as G is socle-regular, it follows from the theorem above that A is strongly socle-regular.

Two Open Problems

Problem 1: If $A = G \oplus H$ is a transitive group and H is separable, is G transitive?

Problem 2: If G is a socle-regular group and $p^\omega G$ is finite, is G necessarily strongly socle-regular?

References

- P.V. Danchev and B. Goldsmith, *On the socles of fully invariant subgroups of Abelian p-groups,* Archiv der Math., to appear.
- P.V. Danchev and B. Goldsmith, *On the socles of characteristic subgroups of Abelian p-groups,* submitted.