

# An Upper Cardinal Bound on Absolute E-Rings

Daniel Herden

University of Münster

Festkolloquium in Honor of Rüdiger Göbel

April 25, 2009

- 1 Introduction
  - The main result
  - Some remarks
  - The combinatorial background
- 2 The proof
  - The outline
  - Proving Step I
  - Open problems

# The main result

$R$  commutative ring with 1.

Definition (Schulz, 1973)

- *evaluation map*  $\delta : \text{End } R^+ \rightarrow R, \varphi \mapsto \varphi(1)$ .
- $R$  *E-ring*  $:\iff \delta$  is ring isomorphism.

Main Theorem (Herden, Shelah, 2008)

*There are no absolute E-rings of cardinality  $\geq \kappa(\omega)$ .*

Main Theorem (Gobel, Herden, Shelah, 2009)

*For every cardinal  $\aleph_0 \leq \kappa < \kappa(\omega)$  there exists an absolute E-ring of cardinality  $\kappa$ .*

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- What does **absolute** mean?

*A property/construction is **absolute** if it is unaffected by generic extensions of the underlying universe  $V$ .*

- What is  $\kappa(\omega)$ ?

*It is the least cardinal  $\kappa$  such that for every  $c : [\kappa]^{<\aleph_0} \longrightarrow 2$  there exist some countable  $X \subseteq \kappa$  and some  $c_X : \omega \longrightarrow 2$  with*

$$c(Y) = c_X(|Y|) \quad \text{for all } Y \in [X]^{<\aleph_0}.$$

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# The combinatorial background

## Definition

- $\mathcal{T} = (T, c)$   *$\lambda$ -colored tree*  $:\Leftrightarrow$ 
  - ▶  $T \subseteq {}^{\omega}>\kappa$  closed under initial segments,
  - ▶  $c : T \rightarrow \lambda$ .
- $f : \mathcal{T}_1 \rightarrow \mathcal{T}_2$  *colored tree homomorphism*  $:\Leftrightarrow$ 
  - ▶  $\eta_1 \triangleleft \eta_2 \implies f(\eta_1) \triangleleft f(\eta_2)$ ,
  - ▶  $\text{lg } f(\eta) = \text{lg } \eta$ ,
  - ▶  $c_2(f(\eta)) = c_1(\eta)$ .

## Theorem (Shelah, 1982)

For any  $\kappa \geq \kappa(\omega)$ ,  $\lambda < \kappa(\omega)$  and any family  $\{\mathcal{T}_\alpha \mid \alpha < \kappa\}$  of  $\lambda$ -colored trees  $\subseteq {}^{\omega}>\kappa$  there exist  $\alpha \neq \beta < \kappa$  with  $\text{Hom}(\mathcal{T}_\alpha, \mathcal{T}_\beta) \neq \emptyset$ .

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## Theorem

*For every abelian group  $A$  of cardinality  $|A| \geq \kappa(\omega)$  there exists a generic extension  $V[G]$  of the universe  $V$  such that*

$$|A| = \aleph_0 \quad \text{and} \quad |\text{End } A| = |\text{Mon } A| = 2^{\aleph_0} \quad \text{in} \quad V[G].$$

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- ▶  $A$  an abelian group  $A$  with  $|A| \geq \kappa(\omega)$ ,
- ▶  $B \subseteq A$  a subgroup with  $|B| < \kappa(\omega)$ ,
- ▶  $V[G]$  a generic extension of the underlying universe  $V$  such that  $|A| = \aleph_0$  holds in  $V[G]$ .

Then: There exists some  $\varphi \in \text{Mon } A$  in  $V[G]$  with

$$\varphi \upharpoonright B = \text{id}_B \quad \text{and} \quad \varphi \neq \text{id}_A.$$

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Let  $A$  be a countable abelian group such that for every finite set  $S \subseteq A$  there exists some  $\varphi \in \text{Mon } A$  with

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# Proving Step I

In V:

- $\bar{s} = \langle s_i | i < m \rangle, \bar{t} = \langle t_j | j < n \rangle \in {}^\omega A,$
- $B_m = B \oplus \bigoplus_{i < m} \mathbb{Z}e_i$  free extension,
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 $\pi_{\bar{s}} \upharpoonright B = \text{id}_B$  and  $\pi_{\bar{s}}(e_i) = s_i$  ( $i < m$ ),
- $\bar{s} \equiv \bar{t} \iff m = n$  and  $\text{Ker } \pi_{\bar{s}} = \text{Ker } \pi_{\bar{t}}.$

Then:

- (1)  $|{}^\omega A / \equiv| \leq \aleph_0 \cdot 2^{|B| + \aleph_0} < \kappa(\omega).$
- (2)  $\bar{s} \equiv \bar{t} \iff n = m$  and there exists an isomorphism  $\psi : \langle B, \bar{s} \rangle \longrightarrow \langle B, \bar{t} \rangle$  with  $\psi \upharpoonright B = \text{id}_B$  and  $\psi(s_i) = t_i$  ( $i < m$ ).

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 $\pi_{\bar{s}} \upharpoonright B = \text{id}_B$  and  $\pi_{\bar{s}}(e_i) = s_i$  ( $i < m$ ),
- $\bar{s} \equiv \bar{t} : \iff m = n$  and  $\text{Ker } \pi_{\bar{s}} = \text{Ker } \pi_{\bar{t}}.$

Then:

- (1)  $|{}^\omega A| \equiv | \leq \aleph_0 \cdot 2^{|B| + \aleph_0} < \kappa(\omega).$
- (2)  $\bar{s} \equiv \bar{t} \iff n = m$  and there exists an isomorphism  $\psi : \langle B, \bar{s} \rangle \longrightarrow \langle B, \bar{t} \rangle$  with  $\psi \upharpoonright B = \text{id}_B$  and  $\psi(s_i) = t_i$  ( $i < m$ ).

# Proving Step I

In V:

- $\bar{s} = \langle s_i | i < m \rangle, \bar{t} = \langle t_j | j < n \rangle \in {}^\omega A,$
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*For every abelian group  $A$  of cardinality  $|A| \geq \kappa(\omega)$  there exists a generic extension  $V[G]$  of the universe  $V$  such that*

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Thank you!