The ϕ_J Polar Decomposition

Joint work in progress with
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Polar Form of a Complex Number

If $z \in \mathbb{C}$, then

$$z = re^{i\theta},$$

for some $r \geq 0$ and $\theta \in \mathbb{R}$.

Polar Decomposition of a Complex Matrix

If $A \in M_n(\mathbb{C})$, then

$$A = PU,$$

for some positive semidefinite P and unitary U.

Consider the function $\gamma : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ defined by

$$\gamma(A) = A^*$$

Then P and U in the polar decomposition of A satisfy

$$\gamma(P) = P$$

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Definition of ϕ_S

Let S be nonsingular and $\phi_S : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ such that

$$\phi_S(A) = S^{-1}A^T S$$

- $\phi_S(rA + tB) = r\phi_S(A) + t\phi_S(B)$
- $\phi_S(AB) = \phi_S(B)\phi_S(A)$
- If A is nonsingular, $\phi_S(A^{-1}) = \phi_S(A)^{-1}$.
- $\phi_S^2 = id_{M_n(\mathbb{C})}$ iff $\phi_S = \phi_K$ for some symmetric or skew-symmetric K.

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- A is said to be ϕ_S **symmetric** if $\phi_S(A) = A$
- A is said to be ϕ_S **orthogonal** if $\phi_S(A) = A^{-1}$
- A is said to **have a ϕ_S polar decomposition** if there exist ϕ_S symmetric X and ϕ_S orthogonal Y such that $A = XY$
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Symmetric or Skew-Symmetric S

(R. Horn & D. Merino, 1995) Let S be nonsingular and symmetric or skew-symmetric.

- If A has a ϕ_S polar decomposition, then $A\phi_S(A)$ is similar to $\phi_S(A)A$.
- If S is symmetric, the necessary condition above is also sufficient for the existence of a polar decomposition.
- Every nonsingular matrix in $M_n(\mathbb{C})$ has a ϕ_S polar decomposition.
- **Problem:** Find necessary and sufficient conditions for singular A to have a ϕ_S polar decomposition when S is skew-symmetric .

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Skew-Symmetric S

- If $S \in M_k(\mathbb{C})$ is nonsingular and skew-symmetric, then
  - k is even, say $k = 2n$
  - $S = PJP^T$, for some nonsingular P and $J \equiv \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$
    - A has a ϕ_S polar decomposition

      iff

      P^TAP^{-T} has a ϕ_J polar decomposition

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Symplectic and Skew-Hamiltonian Matrices

- ϕ_J orthogonal matrices are also known as **symplectic** matrices
- ϕ_J symmetric matrices are also known as **skew-hamiltonian** matrices
- If A has a ϕ_J polar decomposition, then A has even rank

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Symplectic Equivalence

- A is said to be **symplectically equivalent** to B if there exist symplectic P and Q such that $A = PBQ$.
- If $A = PBQ$ for some symplectic P and Q, then
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Reduction

If $A \in M_{2n}(\mathbb{C})$ has even rank $2m$, then A is symplectically equivalent to a matrix of the form

$$\begin{array}{r} k\{ \\ n-k\{ \\ 2m-k\{ \\ n-(2m-k)\{ \end{array} \begin{bmatrix} \overbrace{B}^{i} & \overbrace{0}^{n-i} & \overbrace{C}^{2m-i} & \overbrace{0}^{n-(2m-i)} \\ 0 & 0 & 0 & 0 \\ D & 0 & E & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

for some $2m-k \leq k \leq n$ and $2m-i \leq i \leq n$.

We denote this form by $R(k, 2m-k)C(i, 2m-i)$.

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Some Cases

Let $A \in M_{2n}(\mathbb{C})$ have rank $2m$ and be symplectically equivalent to a matrix of the form $R(k, 2m-k)C(i, 2m-i)$.

- If $i \neq k$, then A has **no** ϕ_J polar decomposition.
- Suppose A has the form $R(k, 2m-k)C(k, 2m-k)$.
  - If $k = m$ or $k = 2m$, then A has a ϕ_J polar decomposition.
  - If $k = 2m-1$, then A has a ϕ_J polar decomposition provided $A\phi_J(A)A$ has even rank.
  - If A has rank at most 4, then A has a ϕ_J polar decomposition **iff** $A\phi_J(A)$ is similar to $\phi_J(A)A$ **and** rank $A\phi_J(A)A$ is even.

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Conjecture

A has a ϕ_J polar decomposition if and only if

- $A\phi_J(A)$ is similar to $\phi_J(A)A$
- $A\phi_J(A)$ has a ϕ_J symmetric square root
- rank $A[\phi_J(A)A]^k$ is even for all $k \geq 0$