# Flat Mittag–Leffler modules and Drinfeld vector bundles

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V.Drinfeld: "Infinite–dimensional vector bundles in algebraic geometry: an introduction," The Unity of Mathematics, Birkhäuser, Boston 2006, 263–304.

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Replace 'projective' by 'flat'.
Then $\mathfrak{F}$ is the flat quasi–coherent sheaf.

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$\mathcal{L}$ the class of all Mittag–Leffler modules, ie., the modules $M$ such that the canonical map $M \otimes_R \prod_{i \in I} M_i \to \prod_{i \in I} M \otimes_R M_i$ is monic for each family of left $R$–modules $(M_i \mid i \in I)$.

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Each module in $\mathcal{P}$ is a direct sum of countably generated modules.

The class $\mathcal{F}$ is not classifiable in case $R$ is not right perfect
(eg., $\mathcal{F}$ = all torsion–free groups in case $R = \mathbb{Z}$).

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(A group is $\aleph_1$–free if each of its countable subgroups is free.)

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- *If $\kappa$ is a regular uncountable cardinal and $M$ is $\kappa$–projective, then $M$ is a flat Mittag–Leffler module.*

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Direct sums of copies of modules from $\mathcal{C}$ are always $\mathcal{C}$–filtered.

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*Let $X$ be a semi–separated scheme and $\kappa$ an infinite cardinal. For each $v \in V$, let $S_v$ be a syzygy closed set of $\leq \kappa$–presented flat $R(v)$–modules such that $M \otimes_{R(v)} N \in S_v$ for all $M, N \in S_v$.*

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**Further cases**

[Asensio–Estrada–Prest–T.] 'Restricted' Drinfeld vector bundles, etc.

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## Corollary

*The homotopy theory tools above apply to vector bundles and flat quasi–coherent sheaves, but* **not** *to Drinfeld vector bundles.*

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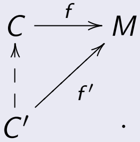

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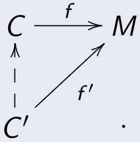


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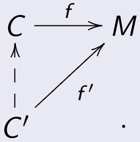


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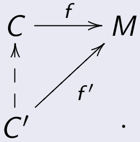


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$$\begin{array}{ccc} C & \xrightarrow{\;f\;} & M \\ \uparrow & \nearrow_{f'} & \\ C' & & . \end{array}$$

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## Theorem (Göbel–T.'00, Eklof–Shelah'03, Asensio–Estrada–Prest–T.)

*The class $\mathcal{D}$ ($=$ all $\aleph_1$–free abelian groups) is not precovering for $R = \mathbb{Z}$.*